

The first law of AdS black holes in higher curvature gravity

Jun-ichirou Koga*

*Advanced Research Institute for Science and Engineering,
Waseda University, Shinjuku-ku, Tokyo 169-8555, Japan*

We consider the first law of black hole thermodynamics in an asymptotically anti-de Sitter spacetime in the class of gravitational theories whose gravitational Lagrangian is an arbitrary function of the Ricci scalar. We first show that the conserved quantities in this class of gravitational theories constructed through conformal completion remain unchanged under the conformal transformation into the Einstein frame. We then prove that the mass and the angular momenta defined by these conserved quantities, along with the entropy defined by the Noether charge, satisfy the first law of black hole thermodynamics, not only in Einstein gravity but also in the higher curvature gravity within the class under consideration. We also point out that it is naturally understood in the symplectic formalism that the mass satisfying the first law should be necessarily defined associated with the timelike Killing vector nonrotating at infinity. Finally, a possible generalization into a wider class of gravitational theories is discussed.

I. INTRODUCTION

Although more than thirty years have passed since the laws of black hole thermodynamics were first formulated [1], it is quite recent that the thermodynamical variables of the Kerr–anti-de Sitter (AdS) black hole in Einstein gravity that satisfy the first law have been correctly identified [2]. Since asymptotically AdS spacetimes are attracting much attention [3] and black hole thermodynamics is expected to play a key role in quantum gravity, it is interesting to explore to what extent black hole thermodynamics is robust in an asymptotically AdS spacetime against quantum gravitational effects. A possible way to proceed in this direction is to consider the effects of higher curvature corrections to Einstein gravity, since such corrections naturally arise due to quantum effects [4, 5].

In the case of an asymptotically flat black hole, black hole thermodynamics is endowed with a reliable foundation, in the sense that the first law has been shown to hold in a quite general context for a stationary and axisymmetric black hole with a bifurcate Killing horizon, without depending on particular solutions or gravitational theories [6]. A bifurcate Killing horizon has a surface where the Killing vector that generates the Killing horizon vanishes, which is called a bifurcation surface, and a bifurcation surface is shown to be possessed by a generic regular Killing horizon with nonvanishing temperature [7]. The first law of an asymptotically flat black hole is then proved [6] in the symplectic formalism (the covariant phase space formalism) by employing Noether charges associated with diffeomorphism invariance. The Noether charge conjugate to the Killing vector that generates the Killing horizon gives the entropy of a black hole multiplied by its temperature when it is evaluated at the Killing horizon, and then one finds that the entropy is not necessarily given by one quarter of its area (in the units $c = G = \hbar = 1$) if higher curvature corrections are included. Thus, black hole entropy is defined based on the first law in the symplectic formalism. At spacial infinity, on the other hand, the Noether charge conjugate to the timelike Killing vector and those conjugate to the axial Killing vectors yield the Arnowitt–Deser–Misner mass and Komar’s angular momenta, respectively. The latter fact is not affected by higher curvature corrections, since these corrections fall off rapidly enough at the asymptotically flat region and hence do not contribute to the mass and the angular momenta.

In the case of an asymptotically AdS spacetime, however, black hole thermodynamics in the general context, as in the asymptotically flat case, has not been established yet. When we focus on Einstein gravity, the mass and the angular momenta of the Kerr–AdS black hole are shown to satisfy the first law of black hole thermodynamics [2], if they are defined by the conserved quantities constructed by Ashtekar, Magnon, and Das through conformal completion [8], which we call conformal charges in this paper, while the mass should be defined associated with the timelike Killing vector nonrotating at infinity. In a more general context, but

*Electronic address: koga@gravity.phys.waseda.ac.jp

still within Einstein gravity, Hollands, Ishibashi and Marolf [9] have recently shown that Noether charges evaluated at infinity of an asymptotically AdS spacetime coincide with conformal charges. As we will see in this paper, this implies that the mass and the angular momenta of a black hole defined by these conformal charges, along with the entropy defined as one quarter of the area of the black hole horizon, satisfy the first law of black hole thermodynamics generally in Einstein gravity, not only for the Kerr–AdS black hole, under suitable assumptions on the asymptotic behavior of matter fields at infinity.

Noether charges are actually defined by their variations in the phase space, not by Noether charges themselves. In order that Noether charges are integrable, one then needs to impose boundary conditions at infinity [10], since otherwise Noether charges, which give the mass and the angular momenta, are not well-defined, which in turn invalidates the first law. While one might expect that the entropy of a black hole is always defined by the Noether charge on the horizon, not only for an asymptotically flat black hole but also for an asymptotically AdS black hole, defining entropy based on the first law does not make sense unless the first law itself is well-defined. It is thus very crucial that boundary conditions are imposed at infinity so that the mass and the angular momenta of a black hole are given by well-defined Noether charges. On the other hand, boundary conditions at infinity are closely related with how an asymptotically AdS spacetime is defined. In order to define an asymptotically AdS spacetime in a rigorous manner, however, it is plausible to employ the conformal completion technique, with which we can construct well-defined conformal charges [8], i.e., integrated conserved quantities, not their variations, without subtracting background contribution or employing a counterterm. Then, it is remarkable that Noether charges at infinity have been proven to coincide with conformal charges in Einstein gravity [9], since it implies that Noether charges that define the mass and the angular momenta and satisfy the first law are indeed well-defined in Einstein gravity.

In higher curvature gravity, an asymptotically AdS spacetime has been defined recently through the conformal completion technique and well-defined conformal charges have been explicitly constructed [11]. Then, it is interesting to investigate whether the mass and the angular momenta defined by these conformal charges satisfy the first law of an asymptotically AdS black hole, not only in Einstein gravity but also in higher curvature gravity. However, it is still unclear whether one can deal with arbitrary theories of gravity at a single stroke. We will thus mainly focus in this paper on the class of gravitational theories whose gravitational Lagrangian is an arbitrary function of the Ricci scalar alone, with the spacetime dimensionality greater than three. It is well-known that such theories of gravity can be conformally transformed into the Einstein frame. We will then begin in Sec. II with a brief review on the definition of an asymptotically AdS spacetime and conformal charges in this class of gravitational theories. We will also analyze how these conformal charges behave under the conformal transformation into the Einstein frame. Based on the result in Sec. II, we will investigate in Sec. III whether the first law of black hole thermodynamics is satisfied when the mass and the angular momenta are defined by these conformal charges and the entropy is defined by the Noether charge on the horizon. In Sec. IV, we will summarize this paper and discuss a possible generalization into a wider class of gravitational theories. Appendix is devoted to the simplest example, where we will calculate the mass and the angular momentum of the four-dimensional Kerr–AdS black hole by using the conformal completion technique in both the original frame and the Einstein frame.

II. CONFORMAL TRANSFORMATION AND CONFORMAL CHARGES

The action of the gravitational theories we consider here is given by

$$I = \int d^n x \sqrt{-g} \left[\frac{f(R)}{16\pi G} + L_m \right], \quad (2.1)$$

where L_m is the Lagrangian of matter fields and the gravitational Lagrangian $f(R)$ is an arbitrary smooth function of the Ricci scalar R . The gravitational field equation that follows from the action Eq. (2.1) is given by

$$f'(R)R_{ab} - \frac{1}{2}g_{ab}f(R) - \nabla_a \nabla_b f'(R) + g_{ab} \nabla_c \nabla^c f'(R) = 8\pi G T_{ab}, \quad (2.2)$$

and thus a spacetime with the Ricci tensor R_{ab} of the form

$$R_{ab} = \frac{R_0}{n} g_{ab}, \quad (2.3)$$

is a solution to Eq. (2.2), if the constant R_0 satisfies

$$f'^{-1}(R_0)f(R_0) = \frac{2}{n}R_0. \quad (2.4)$$

In particular, a spacetime with constant curvature possesses the Ricci tensor of the form of Eq. (2.3), and hence it describes the pure AdS spacetime with the curvature length ℓ , if the constant R_0 is negative and so there exists a real constant ℓ such that

$$R_0 = -\frac{n(n-1)}{\ell^2}. \quad (2.5)$$

A spacetime that asymptotically approaches the pure AdS spacetime is considered as an asymptotically AdS spacetime. By extending the definition in Einstein gravity [8] and imposing the asymptotic condition on the Riemann tensor R_{abcd} , which is regarded as “a boundary condition at infinity”, an asymptotically AdS spacetime is defined in Ref. [11], so that it works also in higher curvature gravity, as

Definition 1 *An n -dimensional spacetime (\mathcal{M}, g_{ab}) with $n \geq 4$ is said to be an asymptotically AdS spacetime if there exists a spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ with a boundary \mathcal{I} equipped with the smooth metric \hat{g}_{ab} , which satisfies following conditions:*

1. $\hat{\mathcal{M}} \setminus \mathcal{I} \cong \mathcal{M}$ and $\mathcal{I} \cong S^{n-2} \times \mathbb{R}$.
2. There exists a smooth scalar Ω on $\hat{\mathcal{M}}$, such that $\hat{g}_{ab} = \Omega^2 g_{ab}$ on \mathcal{M} , and $\Omega = 0$ and $\hat{\nabla}_a \Omega \neq 0$ on \mathcal{I} , where $\hat{\nabla}_a$ is the covariant derivative associated with the unphysical metric \hat{g}_{ab} .
3. The physical metric g_{ab} satisfies the gravitational equation on \mathcal{M} , and there exists a smooth tensor τ_{ab} such that the energy-momentum tensor T_{ab} in a neighborhood of \mathcal{I} is given by $T_{ab} = \Omega^{n-2} \tau_{ab}$.
4. There exist a smooth tensor H_{abcd} and a real constant ℓ such that the Riemann tensor R_{abcd} associated with the physical metric g_{ab} in a neighborhood of \mathcal{I} is given by $R_{abcd} + \ell^{-2} (g_{ac}g_{db} - g_{ad}g_{cb}) = \Omega^{n-5} H_{abcd}$.

In the case of four-dimensional spacetimes, however, we need to impose the so-called reflective boundary condition [8] on the magnetic part of the Weyl tensor in order to obtain a universal group of asymptotic symmetries on \mathcal{I} . Although it will be plausible in future investigations to refine the definition of an asymptotically AdS spacetime to include in a natural manner this exceptional condition in four dimensions, here we simply follow the previous frameworks [8, 11] and impose the reflective boundary condition in this paper. Then, it is shown that \mathcal{I} is conformally flat and thus the asymptotic symmetry group on \mathcal{I} is the n -dimensional AdS group. Furthermore, it has been shown in Ref. [11] that conformal charges $\mathcal{Q}[\xi]$ associated with this asymptotic symmetry group on \mathcal{I} are given, in the class of gravitational theories Eq. (2.1), as

$$\mathcal{Q}[\xi] = -\frac{\ell f'(R_0)}{8\pi G(n-3)} \int_{\mathcal{C}} dx^{n-2} \sqrt{\hat{\sigma}} \hat{\mathcal{E}}_{ab} \xi^a \hat{N}^b, \quad (2.6)$$

on an arbitrary cross section \mathcal{C} of \mathcal{I} , where $\sqrt{\hat{\sigma}}$ is the volume element of \mathcal{C} , \hat{N}^a is the future-pointing timelike unit normal to \mathcal{C} (with respect to the unphysical metric \hat{g}_{ab}), and ξ^a denotes the asymptotic Killing vectors, which generate the asymptotic symmetry group. The argument R_0 of $f'(R_0)$ is the value of the Ricci scalar R on \mathcal{I} , which is given by Eq. (2.5) in terms of the curvature length ℓ at infinity, and the tensor $\hat{\mathcal{E}}_{ab}$ is (the leading part of) the electric part of the Weyl tensor C_{abcd} associated with g_{ab} defined by

$$\hat{\mathcal{E}}_{ab} \equiv \ell^2 \Omega^{5-n} C_{acbd} \hat{n}^c \hat{n}^d, \quad (2.7)$$

where \hat{n}^a is a normal to \mathcal{I} defined as

$$\hat{n}_a \equiv \hat{\nabla}_a \Omega. \quad (2.8)$$

These conformal charges $\mathcal{Q}[\xi]$ are shown to satisfy a physically reasonable form of a balance equation. It is also important to bear in mind that the curvature length ℓ at infinity is determined once the form of

the function $f(R)$ is given, and hence ℓ is fixed and cannot be varied continuously, even when a solution is deformed within the family of asymptotically AdS solutions.

On the other hand, as it is well-known, the conformal transformation

$$g_{ab} \rightarrow \bar{g}_{ab} = \omega^2 g_{ab}, \quad (2.9)$$

where the conformal factor ω is defined by

$$\omega \equiv (f'(R))^{1/(n-2)}, \quad (2.10)$$

and is assumed to be smooth and positive definite everywhere in (\mathcal{M}, g_{ab}) , except for spacetime singularities, transforms the physical spacetime (\mathcal{M}, g_{ab}) in the original frame of the action Eq. (2.1) into the Einstein frame $(\bar{\mathcal{M}}, \bar{g}_{ab})$. The action in the Einstein frame is given by

$$\bar{I} = \int d^n x \sqrt{-\bar{g}} \left[\frac{1}{16\pi G} \{ \bar{R} - (\bar{\nabla}_c \phi) (\bar{\nabla}^c \phi) - U(\phi) \} + (f'(R))^{-n/(n-2)} L_m \right], \quad (2.11)$$

and the extra scalar field ϕ is defined as

$$\phi = \ln(f'(R)) \sqrt{(n-2)/(n-1)}. \quad (2.12)$$

In the Einstein frame, the Ricci scalar R associated with the original metric g_{ab} is considered as a function of ϕ , which is implicitly defined by Eq. (2.12), and is denoted by \mathcal{R} . The potential $U(\phi)$ of ϕ is then found to be given in terms of \mathcal{R} as

$$U(\phi) = (f'(\mathcal{R}))^{-2/(n-2)} \left[\mathcal{R} - (f'(\mathcal{R}))^{-1} f(\mathcal{R}) \right]. \quad (2.13)$$

The field equations in the Einstein frame, apart from those for the ordinary matter fields, are derived as

$$\bar{R}_{ab} - \frac{1}{2} \bar{g}_{ab} \bar{R} + \bar{\Lambda} \bar{g}_{ab} = 8\pi G \bar{T}_{ab}, \quad (2.14)$$

$$2\bar{\nabla}^c \bar{\nabla}_c \phi - U'(\phi) = 16\pi G \frac{n(f'(\mathcal{R}))^{-n/(n-2)}}{\sqrt{(n-1)(n-2)}} L_m. \quad (2.15)$$

In Eq. (2.14), we subtracted out a possible cosmological term $-\bar{\Lambda} \bar{g}_{ab}$ from the matter fields, and thus the energy-momentum tensor \bar{T}_{ab} in the Einstein frame is defined by

$$\bar{T}_{ab} \equiv (f'(\mathcal{R}))^{-1} T_{ab} + \bar{T}_{ab}^{(\phi)} + \frac{\bar{\Lambda}}{8\pi G} \bar{g}_{ab}, \quad (2.16)$$

where $\bar{\Lambda}$ is an effective cosmological constant, which is to be determined below, and the energy-momentum tensor $\bar{T}_{ab}^{(\phi)}$ of the extra scalar field ϕ is given by

$$\bar{T}_{ab}^{(\phi)} = \frac{1}{8\pi G} \left\{ (\bar{\nabla}_a \phi) (\bar{\nabla}_b \phi) - \frac{1}{2} \bar{g}_{ab} [(\bar{\nabla}^c \phi) (\bar{\nabla}_c \phi) + U(\phi)] \right\}. \quad (2.17)$$

If the physical spacetime (\mathcal{M}, g_{ab}) in the original frame is the pure AdS spacetime (with $T_{ab} = 0$), the Ricci scalar R associated with the physical metric g_{ab} in the original frame is given by the constant R_0 that satisfies Eq. (2.4), and thus it is denoted by \mathcal{R}_0 in the Einstein frame. The conformal factor ω and the extra scalar field ϕ are then given by the constants ω_0 and ϕ_0 , which are defined by Eqs. (2.10) and (2.12), respectively, with $R = R_0$ substituted. In this case, the energy-momentum tensor $\bar{T}_{ab}^{(\phi)}$ of the extra scalar field ϕ is given by

$$\bar{T}_{ab}^{(\phi)} = -\frac{1}{16\pi G} \bar{g}_{ab} U(\phi_0) = -\frac{1}{8\pi G} \frac{n-2}{2n} \bar{g}_{ab} (f'(\mathcal{R}_0))^{-2/(n-2)} \mathcal{R}_0. \quad (2.18)$$

By defining the effective cosmological constant $\bar{\Lambda}$ as

$$\bar{\Lambda} \equiv \frac{n-2}{2n} (f'(\mathcal{R}_0))^{-2/(n-2)} \mathcal{R}_0, \quad (2.19)$$

we then see that $\bar{T}_{ab}^{(\phi)}$ is written as

$$\bar{T}_{ab}^{(\phi)} = -\frac{1}{8\pi G} \bar{\Lambda} \bar{g}_{ab}, \quad (2.20)$$

and hence \bar{T}_{ab} vanishes. We also notice that the conformal transformation Eq. (2.9) in this case reduces to rescaling of the metric. If we employ the standard static chart of the pure AdS spacetime, where the metric g_{ab} in the original frame is given by

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega^{n-2}, \quad (2.21)$$

the metric \bar{g}_{ab} in the Einstein frame is written as

$$d\bar{s}^2 = -\left(1 + \frac{\bar{r}^2}{\bar{\ell}^2}\right) d\bar{t}^2 + \left(1 + \frac{\bar{r}^2}{\bar{\ell}^2}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^{n-2}, \quad (2.22)$$

i.e., the same form as Eq. (2.21), while the quantities of the dimension of length are rescaled as

$$\bar{t} \equiv \omega_0 t, \quad \bar{r} \equiv \omega_0 r, \quad \bar{\ell} \equiv \omega_0 \ell. \quad (2.23)$$

We now consider that the physical spacetime (\mathcal{M}, g_{ab}) in the original frame Eq. (2.1) is an asymptotically AdS spacetime, and hence there exists a conformally completed spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$, which satisfies all of the four conditions of Definition 1. We then note that there exists a conformal transformation that brings the physical spacetime $(\mathcal{M}, \bar{g}_{ab})$ in the Einstein frame into the spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$, i.e., the identical spacetime to that obtained from conformal completion of the physical spacetime (\mathcal{M}, g_{ab}) in the original frame by $g_{ab} \rightarrow \hat{g}_{ab} = \Omega^2 g_{ab}$. Actually, we see, by setting as $\bar{\Omega} = \omega^{-1} \Omega$, that the conformal transformation

$$\bar{g}_{ab} \rightarrow \hat{g}_{ab} = \bar{\Omega}^2 \bar{g}_{ab} \quad (2.24)$$

transforms the physical spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the Einstein frame into $(\hat{\mathcal{M}}, \hat{g}_{ab})$, and that $\bar{\Omega} = 0$ and $\hat{\nabla}_a \bar{\Omega} \neq 0$ on \mathcal{I} , because ω is assumed to be smooth and nonvanishing. In addition, since the fourth condition of Definition 1 is satisfied by the physical spacetime (\mathcal{M}, g_{ab}) in the original frame, \mathcal{R} possesses the asymptotic form

$$\mathcal{R} = \mathcal{R}_0 + \bar{\Omega}^{n-1} \mathcal{R}_1, \quad (2.25)$$

and correspondingly the conformal factor ω and the extra scalar field ϕ are written near \mathcal{I} as

$$\omega = \omega_0 + \bar{\Omega}^{n-1} \omega_1, \quad \phi = \phi_0 + \bar{\Omega}^{n-1} \phi_1, \quad (2.26)$$

where \mathcal{R}_1 , ω_1 and ϕ_1 are smooth functions on \mathcal{I} . Since we also have $U'(\phi_0) = 0$, we find from Eqs. (2.17) and (2.26) that $\bar{T}_{ab}^{(\phi)}$ is given as

$$\bar{T}_{ab}^{(\phi)} = -\frac{1}{8\pi G} \bar{\Lambda} \bar{g}_{ab} + O(\bar{\Omega}^{2(n-2)}), \quad (2.27)$$

which implies that $\bar{T}_{ab}^{(\phi)}$ approaches Eq. (2.20) faster than the rate of $\bar{\Omega}^{n-2}$. Furthermore, the energy-momentum tensor T_{ab} in the original frame falls off as Ω^{n-2} due to the third condition of Definition 1 and $(f'(\mathcal{R}))^{-1}$ is smooth at \mathcal{I} . We therefore see that $(f'(\mathcal{R}))^{-1} T_{ab}$ and hence the energy momentum tensor \bar{T}_{ab} in the Einstein frame fall off at the rate of $\bar{\Omega}^{n-2}$. Finally, from Eq. (2.26) and the fourth condition of Definition 1, along with the relation between the Riemann tensor R_{abcd} associated with g_{ab} and \bar{R}_{abcd} associated with \bar{g}_{ab} , we can show that there exist a smooth tensor \bar{H}_{abcd} and a real constant ℓ such that

$$\bar{R}_{abcd} + \frac{1}{\bar{\ell}^2} (\bar{g}_{ac} \bar{g}_{db} - \bar{g}_{ad} \bar{g}_{cb}) = \bar{\Omega}^{n-5} \bar{H}_{abcd}, \quad (2.28)$$

where the curvature length $\bar{\ell}$ at infinity in the Einstein frame is shown to be related with ℓ in the original frame as described by Eq. (2.23). Therefore, we see that all of the four conditions of Definition 1 are

satisfied by the physical spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the Einstein frame, and hence $(\bar{\mathcal{M}}, \bar{g}_{ab})$ is an asymptotically AdS spacetime, if so is the physical spacetime (\mathcal{M}, g_{ab}) in the original frame. The asymptotic infinity of $(\bar{\mathcal{M}}, \bar{g}_{ab})$ then corresponds to the boundary $\bar{\mathcal{I}}$ of $(\hat{\mathcal{M}}, \hat{g}_{ab})$ where $\bar{\Omega}$ vanishes. However, we note here that \mathcal{I} in the original frame and $\bar{\mathcal{I}}$ in the Einstein frame are actually the same boundary of the same spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$. While one can choose different conformal completion [8] by choosing $\bar{\Omega}$ different from $\omega^{-1}\Omega$, there thus exists a mapping with which we can identify an arbitrary cross section $\bar{\mathcal{C}}$ of $\bar{\mathcal{I}}$ in the Einstein frame with the corresponding cross section \mathcal{C} of \mathcal{I} in the original frame.

Once the physical spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the Einstein frame is found to be asymptotically AdS, we need not persist in setting $\bar{\Omega} = \omega^{-1}\Omega$, as we mentioned just above. We then let $\bar{\Omega}$ be given by

$$\bar{\Omega} \equiv \varpi \Omega, \quad (2.29)$$

where ϖ is an arbitrary smooth function that does not vanish on \mathcal{I} , and consider the conformal transformation

$$\bar{g}_{ab} \rightarrow \tilde{g}_{ab} = \bar{\Omega}^2 \bar{g}_{ab}, \quad (2.30)$$

which transforms the physical spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the Einstein frame into a conformally completed spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$. Then, conformal charges $\bar{\mathcal{Q}}[\xi]$ on a cross section $\bar{\mathcal{C}}$ of $\bar{\mathcal{I}}$ in the Einstein frame are given [8] by

$$\bar{\mathcal{Q}}[\xi] = -\frac{\bar{\ell}}{8\pi G(n-3)} \int_{\bar{\mathcal{C}}} dx^{n-2} \sqrt{\bar{\sigma}} \tilde{\mathcal{E}}_{ab} \xi^a \tilde{N}^b, \quad (2.31)$$

where $\sqrt{\bar{\sigma}}$ is the volume element of $\bar{\mathcal{C}}$, \tilde{N}^a is the future-pointing timelike unit normal to $\bar{\mathcal{C}}$ (with respect to \tilde{g}_{ab}), $\tilde{\mathcal{E}}_{ab}$ is defined as

$$\tilde{\mathcal{E}}_{ab} \equiv \bar{\ell}^2 \bar{\Omega}^{5-n} \bar{C}_{acbd} \tilde{n}^c \tilde{n}^d, \quad (2.32)$$

and \tilde{n}_c is given by

$$\tilde{n}_c \equiv \tilde{\nabla}_c \bar{\Omega}. \quad (2.33)$$

The conformal charges $\bar{\mathcal{Q}}[\xi]$ in the Einstein frame also satisfy the balance equation of the same form as that in the original frame. We note also that the asymptotic Killing vectors ξ^a on \mathcal{I} in the original frame are asymptotic Killing vectors on $\bar{\mathcal{I}}$ in the Einstein frame, as well.

Now we compare the conformal charges $\bar{\mathcal{Q}}[\xi]$ in the Einstein frame with $\mathcal{Q}[\xi]$ in the original frame. We first notice from Eqs. (2.8) and (2.33) that \tilde{n}^a is related with \hat{n}^a as $\tilde{n}^a = \varpi^{-1} \omega_0^{-2} \hat{n}^a$ on \mathcal{I} , and then we obtain $\tilde{\mathcal{E}}_{ab} = \varpi^{3-n} \hat{\mathcal{E}}_{ab}$ on \mathcal{I} by using Eq. (2.23). We can also show $\tilde{N}^a = \varpi^{-1} \omega_0^{-1} \hat{N}^a$ and $\sqrt{\bar{\sigma}} = \varpi^{n-2} \omega_0^{n-2} \sqrt{-\hat{\sigma}}$. By substituting these relations into Eq. (2.31) and utilizing the mapping from $\bar{\mathcal{C}}$ to \mathcal{C} , we find that the conformal charges $\bar{\mathcal{Q}}[\xi]$ in the Einstein frame are written as

$$\bar{\mathcal{Q}}[\xi] = -\frac{\ell \omega_0^{n-2}}{8\pi G(n-3)} \int_{\mathcal{C}} dx^{n-2} \sqrt{\hat{\sigma}} \hat{\mathcal{E}}_{ab} \xi^a \hat{N}^b. \quad (2.34)$$

It is interesting to notice here that the conformal charges depend on the conformal factor ω , whereas they do not on ϖ , as they should not [8].

From Eqs. (2.6), (2.10), and (2.34), we therefore find

$$\bar{\mathcal{Q}}[\xi] = \mathcal{Q}[\xi], \quad (2.35)$$

i.e., the conformal charges $\mathcal{Q}[\xi]$ in the original frame and $\bar{\mathcal{Q}}[\xi]$ in the Einstein frame coincide with each other for the same asymptotic Killing vector ξ^a .

III. THE FIRST LAW

We consider in this section the first law of black hole thermodynamics in an asymptotically AdS black hole spacetime in both the original frame and the Einstein frame. We assume here, however, that global charges

independent of the mass and the angular momenta, such as the electromagnetic charge, are not possessed by the black hole, and that the matter fields fall off rapidly enough so that they not only satisfy the third condition of Definition 1, but also make no contribution to Noether charges at infinity, which is actually shown to be true in Ref. [9] under suitable assumptions on the matter fields.

We then consider a stationary and axisymmetric black hole spacetime with a bifurcate Killing horizon in the original frame. Thus, there exist a timelike Killing vector $\xi_{(t)}^a$ and axial Killing vectors $\xi_{(i)}^a$ in the original frame, and the linear combination $\xi_{(h)}^a$ of $\xi_{(t)}^a$ and $\xi_{(i)}^a$ defined by

$$\xi_{(h)}^a \equiv \xi_{(t)}^a + \Omega^{(i)} \xi_{(i)}^a, \quad (3.1)$$

which generates the Killing horizon, vanishes on the bifurcation surface, where $\Omega^{(i)}$ are the angular velocities of the black hole. (There exist $[(n-1)/2]$ axial Killing vectors. The index (i) labels these axial Killing vectors and the corresponding angular velocities, and hence it runs from 1 to $[(n-1)/2]$. Summation over the index (i) is also understood in Eq. (3.1).) However, the Killing vectors $\xi_{(t)}^a$ and $\xi_{(i)}^a$ are not uniquely determined, because any linear combinations of them with constant coefficients are also Killing vectors. In addition, as it has been pointed out by Gibbons et al. [2], one has to take care that the time coordinate nonrotating at infinity with respect to “the AdS background” is used to define the mass of an asymptotically AdS black hole. In a coordinate system where the metric g_{ab} behaves asymptotically as Eq. (2.21), for example, with which one can naturally define “the AdS background”, the mass should be defined associated with the time coordinate t . We therefore set $\xi_{(t)}^a$ and $\xi_{(i)}^a$ so that all of them are orthogonal to each other on \mathcal{I} and the orbits of $\xi_{(i)}^a$ on \mathcal{I} close with the period of 2π . (See also the comment below, where we discuss that the Killing vectors should be necessarily fixed with respect to “the AdS background” in the symplectic formalism.)

We note here that the conformal factor ω is a function only of the Ricci scalar R and the Lie derivatives of R along the Killing vectors $\xi_{(t)}^a$ and $\xi_{(i)}^a$ vanish, which implies that the Lie derivatives of the physical metric \bar{g}_{ab} in the Einstein frame along these vectors vanish, as well. Therefore, there exist Killing vectors $\bar{\xi}_{(t)}^a$ and $\bar{\xi}_{(i)}^a$ also in the Einstein frame, which are given by constant multiples of $\xi_{(t)}^a$ and $\xi_{(i)}^a$, respectively. We then normalize the axial Killing vectors $\bar{\xi}_{(i)}^a$ as

$$\xi_{(i)}^a = \bar{\xi}_{(i)}^a. \quad (3.2)$$

On the other hand, we set the relative normalization between $\xi_{(t)}^a$ and $\bar{\xi}_{(t)}^a$ as

$$\bar{\xi}_{(t)}^a = \omega_0^{-1} \xi_{(t)}^a, \quad (3.3)$$

in accordance with the rescaling property described by Eq. (2.23). In a coordinate system that behaves asymptotically as Eq. (2.21), we then normalize the timelike Killing vector $\xi_{(t)}^a$ in the original frame as $\xi_{(t)}^a = (\partial/\partial t)^a$, and thus the timelike Killing vector $\bar{\xi}_{(t)}^a$ in the Einstein frame is expressed as $\bar{\xi}_{(t)}^a = (\partial/\partial \bar{t})^a$, where \bar{t} is the time coordinate in Eq. (2.22).

Since the Killing vectors $\xi_{(t)}^a$ and $\xi_{(i)}^a$, and hence $\bar{\xi}_{(t)}^a$ and $\bar{\xi}_{(i)}^a$, are asymptotic Killing vectors, we can define the conformal charges $\mathcal{Q}[\xi_{(t)}]$ and $\mathcal{Q}[\xi_{(i)}]$ associated with $\xi_{(t)}^a$ and $\xi_{(i)}^a$ in the original frame (\mathcal{M}, g_{ab}) , and $\bar{\mathcal{Q}}[\bar{\xi}_{(t)}]$ and $\bar{\mathcal{Q}}[\bar{\xi}_{(i)}]$ associated with $\bar{\xi}_{(t)}^a$ and $\bar{\xi}_{(i)}^a$ in the Einstein frame $(\bar{\mathcal{M}}, \bar{g}_{ab})$. By using Eqs. (2.35), (3.2), and (3.3), as well as the fact that $\mathcal{Q}[\xi]$ is linear in ξ^a for arbitrary asymptotic Killing vectors ξ^a , we then obtain

$$\mathcal{Q}[\xi_{(t)}] = \omega_0 \bar{\mathcal{Q}}[\bar{\xi}_{(t)}], \quad \mathcal{Q}[\xi_{(i)}] = \bar{\mathcal{Q}}[\bar{\xi}_{(i)}]. \quad (3.4)$$

The mass \bar{M} and the angular momenta $\bar{J}_{(i)}$ in the Einstein frame are defined, as if in Einstein gravity, by the conformal charges $\mathcal{Q}[\xi]$ as

$$\bar{M} \equiv \bar{\mathcal{Q}}[\bar{\xi}_{(t)}], \quad \bar{J}_{(i)} \equiv -\bar{\mathcal{Q}}[\bar{\xi}_{(i)}]. \quad (3.5)$$

We will show in Appendix that Eq. (3.5) correctly yields the mass and the angular momentum of the four-dimensional Kerr–AdS black hole in Einstein gravity [2], while the sign of the second equation in Eq. (3.5)

is opposite to that presented in Ref. [2]. (See the comment at the end of Appendix below.) We then define the mass M and the angular momenta $J_{(i)}$ of the black hole in the original frame also by

$$M \equiv \mathcal{Q}[\xi_{(t)}], \quad J_{(i)} \equiv -\mathcal{Q}[\xi_{(i)}]. \quad (3.6)$$

Actually, one can show that the definition of the mass in Eq. (3.6) reproduces the thermodynamical energy derived in Ref. [12] for spherically symmetric black holes in the class of gravitational theories Eq. (2.1). (It has been shown that Eq. (3.6) gives the correct mass also in Einstein–Gauss–Bonnet gravity [11].) From Eqs. (3.4), (3.5) and (3.6), we therefore find that the masses and the angular momenta in the two frames are related as

$$M = \omega_0 \bar{M}, \quad J_{(i)} = \bar{J}_{(i)}. \quad (3.7)$$

Now we recall the fact that a conformal transformation does not alter the causal structure of a spacetime. In addition, we can show that there exists a Killing vector, which becomes null on the hypersurface in the Einstein frame that corresponds to the Killing horizon in the original frame. Indeed, by defining the angular velocities $\bar{\Omega}^{(i)}$ in the Einstein frame by

$$\bar{\Omega}^{(i)} \equiv \omega_0^{-1} \Omega^{(i)}, \quad (3.8)$$

we find that the Killing vector $\bar{\xi}_{(h)}^a$ defined by

$$\bar{\xi}_{(h)}^a \equiv \bar{\xi}_{(t)}^a + \bar{\Omega}^{(i)} \bar{\xi}_{(i)}^a = \omega_0^{-1} \xi_{(h)}^a, \quad (3.9)$$

is null on the hypersurface in the Einstein frame where the Killing vector $\xi_{(h)}^a$ defined by Eq. (3.1) becomes null in the original frame. Moreover, as we see from Eqs. (3.1) and (3.9), the bifurcation surface in the original frame is a bifurcation surface also in the Einstein frame, because $\bar{\xi}_{(h)}^a$ vanishes when $\xi_{(h)}^a$ does. Therefore, the physical spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the Einstein frame is found to be a black hole spacetime with a bifurcate Killing horizon, if so is the physical spacetime (\mathcal{M}, g_{ab}) in the original frame, and the bifurcation surface appears at the same location in both the frames. We also find that the rescaling of the timelike Killing vector Eq. (3.3) results in the rescaling of the surface gravity of the black hole. The surface gravity κ in the original frame and $\bar{\kappa}$ in the Einstein frame, which are defined by

$$\xi_{(t)}^b \nabla_b \xi_{(t)}^a \equiv \kappa \xi_{(t)}^a, \quad \bar{\xi}_{(t)}^b \bar{\nabla}_b \bar{\xi}_{(t)}^a \equiv \bar{\kappa} \bar{\xi}_{(t)}^a \quad (3.10)$$

on the horizon, are related as

$$\bar{\kappa} = \omega_0^{-1} \kappa, \quad (3.11)$$

where ∇_b and $\bar{\nabla}_b$ are the covariant derivatives associated with g_{ab} and \bar{g}_{ab} , respectively.

One may expect that black hole entropy is defined without being affected by the geometric structure at infinity, since it is evaluated quasi-locally only on the horizon. This suggests that the entropy of an asymptotically AdS black hole is defined in the same manner as that of an asymptotically flat black hole [6]. Thus, by using the Noether charge conjugate to $\xi_{(h)}^a$ evaluated at the bifurcation surface H , which we denote by $Q_H[\xi_{(h)}]$, we define the entropy S of the black hole in the original frame by

$$Q_H[\xi_{(h)}] \equiv \frac{\kappa}{2\pi} S, \quad (3.12)$$

where $Q_H[\xi_{(h)}]$ is calculated from the action Eq. (2.1) as

$$Q_H[\xi_{(h)}] = \frac{\kappa}{8\pi G} \int_H dx^{n-2} \sqrt{\sigma} f'(R). \quad (3.13)$$

In the case of an asymptotically flat black hole, it has been proven [13, 14] that the entropy of a black hole in the original frame and that in the Einstein frame coincide with each other. It occurs essentially because the Noether charge $Q_H[\xi_{(h)}]$ in the original frame coincides with $\bar{Q}_H[\xi_{(h)}]$ in the Einstein frame for the same Killing vector $\xi_{(h)}^a$. Since the Noether charge $Q_H[\xi_{(h)}]$ depends only on tensors at the horizon, we can repeat

exactly the same argument also in the case of an asymptotically AdS black hole. Actually, from Eqs. (2.9) and (2.10), we obtain

$$\bar{Q}_H[\xi_{(h)}] = \frac{\kappa}{8\pi G} \int_H dx^{n-2} \sqrt{\bar{\sigma}} = \frac{\kappa}{8\pi G} \int_H dx^{n-2} \sqrt{\sigma} f'(R), \quad (3.14)$$

and hence we have

$$Q_H[\xi_{(h)}] = \bar{Q}_H[\xi_{(h)}] \quad (3.15)$$

also for an asymptotically AdS black hole. However, as we see from Eq. (3.10), the surface gravity $\bar{\kappa}$ in the Einstein frame is defined with respect to $\bar{\xi}_{(h)}^a$, not $\xi_{(h)}^a$, and therefore the entropy \bar{S} in the Einstein frame should be defined by the Noether charge $\bar{Q}_H[\bar{\xi}_{(h)}]$ conjugate to $\bar{\xi}_{(h)}^a$ as

$$\bar{Q}_H[\bar{\xi}_{(h)}] \equiv \frac{\bar{\kappa}}{2\pi} \bar{S}. \quad (3.16)$$

Since the Noether charge $\bar{Q}_H[\bar{\xi}_{(h)}]$ is linear in $\bar{\xi}_{(h)}^a$, we see from Eqs. (3.3), (3.11), (3.12), (3.15), and (3.16) that the entropy S in the original frame and \bar{S} in the Einstein frame coincide with each other, i.e.,

$$\bar{S} = S. \quad (3.17)$$

For an arbitrary gravitational theory with diffeomorphism invariance, it is shown in the symplectic formalism [6] that the variations of Noether charges conjugate to arbitrary Killing vectors ξ^a satisfy the spacial conservation law as

$$\delta Q_H[\xi] = \delta Q_{\mathcal{I}}[\xi], \quad (3.18)$$

where $Q_{\mathcal{I}}[\xi]$ denotes the Noether charges evaluated at infinity. It is crucial here to notice that Eq. (3.18) is valid whatever the asymptotic structure of the spacetime at infinity is, as long as the Noether charges are well-defined. Then, if the Noether charges are indeed integrable, the first law of black hole thermodynamics

$$\frac{\kappa}{2\pi} \delta S = \delta M - \Omega^{(i)} \delta J_{(i)} \quad (3.19)$$

follows by setting as $\xi^a = \xi_{(h)}^a$ and defining the mass M , the angular momenta $J_{(i)}$, and the entropy S by

$$M \equiv Q_{\mathcal{I}}[\xi_{(t)}], \quad J_{(i)} \equiv -Q_{\mathcal{I}}[\xi_{(i)}], \quad S \equiv \frac{2\pi}{\kappa} Q_H[\xi_{(h)}], \quad (3.20)$$

where we note that the variations of the surface gravity vanish when evaluated on the bifurcation surface and that $\delta Q_{\mathcal{I}}[\xi_{(h)}]$ is linear in $\xi_{(h)}^a$.

It is worth mentioning here that Killing vectors ξ^a , to which Noether charges are conjugate, must be kept fixed under the variations that appear in Eq. (3.18). This is because these variations are actually identical to those in the variational principle of the action. Therefore, they act only on dynamical fields, such as the metric and matter fields, but should not act on the Killing vectors. On the other hand, these variations are subject to boundary conditions at infinity, by definition. (Recall the fact that boundary conditions should be imposed so that Noether charges are ensured to be integrable. This implies that the variations of Noether charges are subject to the boundary conditions, since otherwise Noether charges are not well-defined.) These facts imply that the Killing vectors ξ^a should remain fixed with respect to the universal structure determined by the boundary conditions at infinity. In the case of an asymptotically AdS black hole, in particular, one should necessarily use the timelike Killing vector that is fixed with respect to “the AdS background” in order to define the mass by Eq. (3.20). Thus, it is naturally understood in the symplectic formalism why the mass satisfying the first law must be defined associated with the timelike Killing vector nonrotating at infinity.

In Einstein gravity, it has been also shown that Noether charges evaluated at infinity of an asymptotically AdS spacetime coincide with conformal charges [9]. More precisely, we have

$$\delta \mathcal{Q}[\xi] = \delta Q_{\mathcal{I}}[\xi], \quad (3.21)$$

for arbitrary asymptotic Killing vectors ξ^a , when matter fields fall off rapidly enough at infinity. In particular, a scalar field with the canonical kinetic term is shown [9] to make no contribution to $\delta Q_{\mathcal{I}}[\xi]$ if it falls

off as $\Omega^{(n-1)/2}$. We emphasize here that Eq. (3.21) implies that Noether charges $Q_{\mathcal{I}}[\xi]$ are well-defined, since conformal charges $\bar{Q}[\xi]$ are integrable by definition. One may notice, however, that the definition of an asymptotically AdS spacetime in Ref. [9] differs from Definition 1 in this paper. Actually, an asymptotically AdS spacetime is defined in Ref. [9] by imposing the condition that the induced metric on \mathcal{I} is conformal to that of the Einstein static universe, instead of imposing an asymptotic condition on the curvature tensor. However, as we mentioned before, the asymptotic conditions of Definition 1, along with the reflective boundary condition in four dimensions, imply that \mathcal{I} is conformally flat. Since a conformally flat spacetime is conformal to the Einstein static universe, we see that Eq. (3.21) indeed holds under the conditions of Definition 1.

In the Einstein frame Eq. (2.11), the gravitational field is described by Einstein gravity, while transformed from the original action Eq. (2.1), and the extra scalar field ϕ is found from Eq. (2.26) to fall off faster than $\bar{\Omega}^{(n-1)/2}$ when the effective cosmological constant is subtracted. Therefore, Eq. (3.21) holds even in the Einstein frame Eq. (2.11), and we integrate it as $\bar{Q}[\xi] = \bar{Q}_{\mathcal{I}}[\xi]$ by setting the integration constant so that it vanishes for the pure AdS spacetime. The mass \bar{M} and the angular momenta $\bar{J}_{(i)}$ in the Einstein frame defined by Eq. (3.5) are therefore found to be given by the Noether charges as

$$\bar{M} \equiv \bar{Q}_{\mathcal{I}}[\bar{\xi}_{(t)}], \quad \bar{J}_{(i)} \equiv -\bar{Q}_{\mathcal{I}}[\bar{\xi}_{(i)}]. \quad (3.22)$$

Moreover, the entropy \bar{S} in the Einstein frame is defined by the Noether charge as Eq. (3.16), and hence we see that the first law of black hole thermodynamics is satisfied in the Einstein frame by the mass \bar{M} , the angular momenta $\bar{J}_{(i)}$, and the entropy \bar{S} as

$$\frac{\bar{\kappa}}{2\pi} \delta \bar{S} = \delta \bar{M} - \bar{\Omega}^{(i)} \delta \bar{J}_{(i)}. \quad (3.23)$$

We further note that the value ω_0 of the conformal factor ω at infinity, which depends only on the curvature length ℓ at infinity, does not change even when the mass and the angular momenta are varied. We thus find from Eqs. (3.7), (3.8), (3.11), (3.17), and (3.23), that the first law is satisfied also in the original frame as

$$\frac{\kappa}{2\pi} \delta S = \delta M - \Omega^{(i)} \delta J_{(i)}. \quad (3.24)$$

IV. SUMMARY AND DISCUSSION

We considered the class of gravitational theories described by the action Eq. (2.1), and proved first that conformal charges, i.e., well-defined conserved quantities constructed through conformal completion, in an asymptotically AdS spacetime remain untouched under the conformal transformation into the Einstein frame. We then considered a stationary and axisymmetric asymptotically AdS black hole, and defined the mass and the angular momenta of the black hole by these conformal charges as Eqs. (3.5) and (3.6) in both the original frame and the Einstein frame. Then, by taking into account the rescaling of the timelike Killing vector, the behavior of the thermodynamical variables under the conformal transformation into the Einstein frame was derived. In particular, we found that the entropies in the two frames coincide with each other, as in the asymptotically flat case [13]. By noticing that the result of Ref. [9] implies that the first law of black hole thermodynamics is satisfied in the Einstein frame, it was shown that the first law is satisfied also in the original frame, when the mass and the angular momenta are defined by the conformal charges and the entropy is defined by the Noether charge in the same manner as in the case of an asymptotically flat black hole. We also pointed out that it is naturally understood in the symplectic formalism that the mass satisfying the first law should be necessarily defined associated with the timelike Killing vector that is fixed with respect to “the AdS background”, i.e., the universal structure determined by the boundary conditions at infinity.

The action Eq. (2.1) describes Einstein gravity when $f(R) = R$. Hence, the result in this paper shows, in particular, that the first law of an asymptotically AdS black hole in Einstein gravity is generally satisfied if the mass and the angular momenta are defined by the conformal charges associated with the Killing vectors that are fixed with respect to “the AdS background”. (See also Ref. [15] for another derivation of the first law in Einstein gravity.) This is true also for higher curvature gravity within the class of Eq. (2.1). Actually, it is not difficult to generalize this result further. It is known [16] that the gravitational theories whose action

is given by

$$I = \int d^n x \sqrt{-g} \left[\frac{f(g^{ab}, R_{ab})}{16\pi G} + L_m \right], \quad (4.1)$$

where $f(g^{ab}, R_{ab})$ is a nonsingular function of the metric g^{ab} and the Ricci tensor R_{ab} , are transformed into the Einstein frame by the Legendre transformation

$$g^{ab} \rightarrow \bar{g}^{ab} \equiv \left[-\det \frac{\partial(\sqrt{-g}f(g^{cd}, R_{cd}))}{\partial R_{ef}} \right]^{-1/(n-2)} \frac{\partial(\sqrt{-g}f(g^{cd}, R_{cd}))}{\partial R_{ab}}. \quad (4.2)$$

In this case, the metric \bar{g}^{ab} in the Einstein frame is not necessarily proportional to the metric g^{ab} in the original frame, in contrast to the case of Eq. (2.1). However, \bar{g}^{ab} is constructed only from g^{ab} and R_{ab} . Moreover, R_{ab} is proportional to g_{ab} when the spacetime (\mathcal{M}, g_{ab}) in the original frame is the pure AdS spacetime. Therefore, in the pure AdS spacetime, \bar{g}^{ab} is proportional to g^{ab} , and hence the Legendre transformation Eq. (4.2) reduces to a conformal transformation. This implies that in an asymptotically AdS spacetime, where the Ricci tensor R_{ab} is asymptotically proportional to the metric g_{ab} , the Legendre transformation asymptotically approaches the conformal transformation. Thus, as we can expect from the result in Sec. II, it will be possible to show that conformal charges in an asymptotically AdS spacetime are invariant under the transformation into the Einstein frame even in the case of Eq. (4.1). In addition, it has been proven [14] that the Noether charge $Q_H[\xi_{(h)}]$ conjugate to the Killing vector $\xi_{(h)}^a$, which gives the entropy of a black hole, also remains unchanged under the Legendre transformation. Then, we can employ the same method as shown in Sec. III in order to prove that the mass M and the angular momenta $J_{(i)}$ defined by Eq. (3.6), along with the entropy S defined by Eq. (3.12), satisfy the first law of an asymptotically AdS black hole even in the class of gravitational theories Eq. (4.1).

However, in order to prove the first law for general theories of gravity, which may not be transformed into the Einstein frame, it actually looks more systematic to consider the Noether charges $Q_{\mathcal{I}}[\xi_{(t)}]$ and $Q_{\mathcal{I}}[\xi_{(i)}]$, rather than the conformal charges $Q[\xi_{(t)}]$ and $Q[\xi_{(i)}]$, and analyze the integrability condition of these Noether charges. In addition, the asymptotic condition on matter fields at infinity also should be analyzed in a systematic manner, while we simply assumed in this paper that they fall off rapidly enough.

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APPENDIX A: FOUR-DIMENSIONAL KERR-ADS BLACK HOLE

The Kerr-AdS black hole is a solution in the class of gravitational theories Eq. (2.1) if there exists a negative constant R_0 that satisfies Eq. (2.4), since the Ricci tensor R_{ab} of the Kerr-AdS black hole spacetime takes the form of Eq. (2.3). Then, in this appendix, we calculate the mass and the angular momentum of the four-dimensional Kerr-AdS black hole in both of the original frame and the Einstein frame by using Eqs. (2.6) and (2.31).

We begin with the metric of the Kerr-AdS black hole in the coordinate system (t', r, θ, ϕ') that is *rotating* at infinity,

$$ds^2 = -\frac{\Delta}{\Sigma} \left[dt' - \frac{a}{\Xi} \sin^2 \theta d\phi' \right]^2 + \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left[\frac{r^2 + a^2}{\Xi} d\phi' - a dt' \right]^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2, \quad (A1)$$

where

$$\Delta \equiv (r^2 + a^2) \left(1 + \frac{r^2}{l^2} \right) - 2Gmr, \quad \Delta_\theta \equiv 1 - \frac{a^2}{l^2} \cos^2 \theta, \quad \Sigma \equiv r^2 + a^2 \cos^2 \theta, \quad \Xi \equiv 1 - \frac{a^2}{l^2}. \quad (A2)$$

We then transform Eq. (A1) into the nonrotating coordinate system (t, r, θ, ϕ) by the coordinate transformation

$$t \equiv t', \quad \phi \equiv \phi' + \frac{a}{l^2} t', \quad (A3)$$

where the metric is now written as

$$ds^2 = -\frac{\Delta}{\Sigma\Xi^2} [\Delta_\theta dt - a \sin^2 \theta d\phi]^2 + \frac{\Delta_\theta \sin^2 \theta}{\Sigma\Xi^2} \left[(r^2 + a^2) d\phi - a \left(1 + \frac{r^2}{l^2} \right) dt \right]^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2, \quad (\text{A4})$$

and the angular coordinates range as $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$. Here we note that the timelike Killing vector $\xi_{(t)}^a$ and the axial Killing vector $\xi_{(\phi)}^a$ defined in the nonrotating coordinate system by

$$\xi_{(t)}^a \equiv \left(\frac{\partial}{\partial t} \right)^a, \quad \xi_{(\phi)}^a \equiv \left(\frac{\partial}{\partial \phi} \right)^a \quad (\text{A5})$$

are expressed, respectively, in the rotating coordinate system as

$$\xi_{(t)}^a = \left(\frac{\partial}{\partial t'} \right)^a - \frac{a}{l^2} \left(\frac{\partial}{\partial \phi'} \right)^a, \quad \xi_{(\phi)}^a = \left(\frac{\partial}{\partial \phi'} \right)^a. \quad (\text{A6})$$

Now we assume that the physical metric g_{ab} in the original frame is given by Eq. (A4). We then calculate the mass and the angular momentum of the black hole in the original frame from Eq. (2.6) with $n = 4$. Although a coordinate transformation of r and θ is necessary in order to obtain the metric that behaves asymptotically as Eq. (2.21), we can choose r^{-1} as the conformal factor Ω since r^{-1} is shown to vanish on \mathcal{I} . Thus, by transforming the radial coordinate from r to $\Omega \equiv r^{-1}$, we find that the unphysical metric \hat{g}_{ab} in the original frame is given on \mathcal{I} ($\Omega = 0$) as

$$d\hat{s}^2 = -\frac{\Delta_\theta}{l^2\Xi} dt^2 + l^2 d\Omega^2 + \frac{1}{\Delta_\theta} d\theta^2 + \frac{1}{\Xi} \sin^2 \theta d\phi^2, \quad (\text{A7})$$

and then the normal \hat{n}^a to \mathcal{I} , the future-pointing timelike unit normal \hat{N}^a to \mathcal{C} , and the volume element $\sqrt{\hat{\sigma}}$ of \mathcal{C} are found to be given by

$$\hat{n}^a = \frac{1}{l^2} \left(\frac{\partial}{\partial \Omega} \right)^a, \quad \hat{N}^a = l \sqrt{\frac{\Xi}{\Delta_\theta}} \left(\frac{\partial}{\partial t} \right)^a, \quad \sqrt{\hat{\sigma}} = \frac{1}{\sqrt{\Xi\Delta_\theta}} \sin \theta. \quad (\text{A8})$$

Since the relevant components of the Weyl tensor C_{abcd} associated with the physical metric g_{ab} are calculated as

$$C_{trtr} = -\frac{Gmr\Delta_\theta}{\Xi^2\Sigma^3} \left[2\Delta_\theta + \frac{a^2 \sin^2 \theta}{\Delta} \left(1 + \frac{r^2}{l^2} \right)^2 \right] (r^2 - 3a^2 \cos^2 \theta), \quad (\text{A9})$$

$$C_{\phi rtr} = \frac{Gmar\Delta_\theta \sin^2 \theta}{\Xi^2\Sigma^3} \left[2 + \frac{1}{\Delta} \left(1 + \frac{r^2}{l^2} \right) (r^2 + a^2) \right] (r^2 - 3a^2 \cos^2 \theta), \quad (\text{A10})$$

we obtain the $t-t$ and $\phi-t$ components of the tensor $\hat{\mathcal{E}}_{ab}$ as

$$\hat{\mathcal{E}}_{tt} = -\frac{Gm\Delta_\theta}{l^2\Xi^2} \left[2 + \frac{a^2}{l^2} (1 - 3 \cos^2 \theta) \right], \quad \hat{\mathcal{E}}_{\phi t} = \frac{Gma\Delta_\theta}{l^2\Xi^2} [2 + (1 - 3 \cos^2 \theta)]. \quad (\text{A11})$$

From Eqs. (2.6), (2.10), (A8), and (A11), we see that the mass M and the angular momentum J in the original frame, which are defined by Eq. (3.6), are given as

$$M \equiv \mathcal{Q}[\xi_{(t)}] = \omega_0^2 \frac{m}{\Xi^2}, \quad J \equiv -\mathcal{Q}[\xi_{(\phi)}] = \omega_0^2 \frac{ma}{\Xi^2}. \quad (\text{A12})$$

To calculate the mass and the angular momentum in the Einstein frame, we notice here that the conformal factor ω is constant in this case, which we thus denote as ω_0 , since the Ricci scalar R of the Kerr-AdS black hole spacetime is constant. The physical metric \bar{g}_{ab} in the Einstein frame is then found to be given by the same form as Eq. (A4), while t , r , a , ℓ , and m are replaced by

$$\bar{t} \equiv \omega_0 t, \quad \bar{r} \equiv \omega_0 r, \quad \bar{a} \equiv \omega_0 a, \quad \bar{\ell} \equiv \omega_0 \ell, \quad \bar{m} \equiv \omega_0 m, \quad (\text{A13})$$

respectively, which is understood from the rescaling property Eq. (2.23) at infinity. Then, \hat{n}^a , \hat{N}^a , $\sqrt{\hat{\sigma}}$, and $\hat{\mathcal{E}}_{ab}$, which appear in Eq. (2.31), take the same forms as those given in Eqs. (A8) and (A11), but with the unbarred quantities replaced by the barred ones. The mass \bar{M} and the angular momentum \bar{J} in the Einstein frame defined by Eq. (3.5) are therefore computed from Eqs. (2.31) and (A13) as

$$\bar{M} \equiv \bar{Q}[\bar{\xi}_{(t)}] = \frac{\bar{m}}{\Xi^2} = \omega_0 \frac{m}{\Xi^2}, \quad \bar{J} \equiv -\bar{Q}[\bar{\xi}_{(\phi)}] = \frac{\bar{m}\bar{a}}{\Xi^2} = \omega_0^2 \frac{ma}{\Xi^2}. \quad (\text{A14})$$

We thus see that the mass and the angular momentum of the four-dimensional Kerr–AdS black hole in Einstein gravity [2, 17] are correctly reproduced, i.e., $\omega_0 = 1$ in Eqs. (A12) and (A14) or the barred expression in Eq. (A14), and also that Eq. (3.7) is indeed satisfied.

Finally, we note that the sign of the second equation in Eq. (3.5) and the sign of the second term of the first equation in Eq. (A6) are opposite to those presented in Ref. [2]. However, these two signs cancel with each other in the calculation performed in Ref. [2], and hence the eventual conclusion that the mass defined by Eq. (3.5) satisfies the first law of black hole thermodynamics is unaltered.

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